Calculus 12: Review Part I

1. Evaluate the following limits: (Show your work)

(a) $\lim_{x \to 3} \dfrac{x^2 - 6x + 9}{x - 3}$

(b) $\lim_{x \to -4} \dfrac{2x + 8}{x^2 + x - 12}$

(c) $\lim_{x \to \infty} \dfrac{3x + 5}{6x - 8}$

(d) $\lim_{x \to -\infty} \dfrac{4x^2 - x}{2x^3 - 5}$

(e) $\lim_{x \to 0} \dfrac{6x - 9}{x^3 - 12x + 3}$

(f) $\lim_{x \to \infty} \dfrac{\sqrt{x^2 + 2}}{3x - 6}$

(g) $\lim_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}$

2. Use the definition of derivative to find the derivative of the following:

(a) $f(x) = \sqrt{5x}$

(b) $f(x) = x^3 + x^2$

(c) $f(x) = \sqrt{x+1}$

(d) $f(x) = \frac{1}{x^2}$

3. Determine the constant a so that the following function is continuous on the entire real line

$$f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$$

4. Find the average rate of change for each of the functions over the given intervals:

(a) $y = x^2 + 1$ $[3, 5]$ (b) $f(x) = \frac{1}{x+1}$ $[0, 3]$

5. Find the equations of the lines that are:
(i) tangent and (ii) normal to the curve

(a) $y = 5x^2$ at the point $(-1, 5)$

(b) $y = x^2 - 4x$ at the point $(1, -3)$

6. Using the symmetric difference quotient $\frac{f(a+h) - f(a-h)}{2h}$

Find : $f'(10)$ when $f(x) = x^3$ (Let $h = 0.01$)

7. Find the first derivative of each of the following using the rules for differentiation.

a) $y = 5x^4 - 3x + \frac{4}{x} + 3$

b) $y = \frac{3x}{2x+1}$

c) $y = (2x^2 + 2)\sqrt{x}$

d) $xy = \cos(x+y)$

e) $y = x^3 e^x + xe^{3x}$

f) $y = 2^{5x}$

g) $y = \log_3(\sqrt[3]{x})$

h) $y = \log_7(4 + x\ln 7)$

Evaluate the following limits: (Show your work)

(a) $\lim_{x\to 3} \frac{x^2-6x+9}{x-3}$

$\lim_{x\to 3} \frac{\cancel{(x-3)}(x-3)}{\cancel{(x-3)}} = \lim_{x\to 3} (3)-3 = \boxed{0}$

(b) $\lim_{x\to -4} \frac{2x+8}{x^2+x-12}$

$= \lim_{x\to -4} \frac{2\cancel{(x+4)}}{\cancel{(x+4)}(x-3)} = \lim_{x\to -4} \frac{2}{x-3}$

$= \frac{2}{-4-3} = \boxed{-\frac{2}{7}}$

(c) $\lim_{x\to \infty} \frac{3x+5}{6x-8}$

$= \lim_{x\to \infty} \frac{\frac{3x}{x}+\frac{5}{x}}{\frac{6x}{x}-\frac{8}{x}}$

$= \lim_{x\to \infty} \frac{3+5/x}{6-8/x} = \boxed{\frac{1}{2}}$

(d) $\lim_{x\to -\infty} \frac{4x^2-x}{2x^3-5}$

$= \lim_{x\to -\infty} \frac{\frac{4x^2}{x^3}-\frac{x}{x^3}}{\frac{2x^3}{x^3}-\frac{5}{x^3}} = \lim_{x\to -\infty} \frac{4/x - 1/x^2}{2-5/x^3} = \frac{0}{2} = \boxed{0}$

OR $\frac{4x^2}{2x^3} = \frac{4}{x} \Rightarrow \lim_{x\to -\infty} \frac{4}{x} = 0$

(e) $\lim_{x\to 0} \frac{6x-9}{x^3-12x+3}$

$\frac{6(0)-9}{0^3-12(0)+3} = \frac{-9}{3}$

$= \boxed{-3}$

(f) $\lim_{x\to \infty} \frac{\sqrt{x^2+2}}{3x-6}$

$\lim_{x\to \infty} \frac{\sqrt{1+2/x^2}}{3-6/x} = \boxed{\frac{1}{3}}$

(g) $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$ $\frac{(\sqrt{x+4}+2)}{(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{x+4-4}{x\sqrt{x+4}+2} = \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \boxed{\frac{1}{4}}$

$$f(x)=\sqrt{5x} \qquad \frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{5x+5h}-\sqrt{5x}}{h}\times\left(\frac{\sqrt{5x+5h}+\sqrt{5x}}{\sqrt{5x+5h}+\sqrt{5x}}\right)$$

$$=\lim_{h\to 0}\frac{5x+5h-5x}{h\sqrt{5x+5h}+\sqrt{5x}}=\lim_{h\to 0}\frac{5}{\sqrt{5x+5h}+\sqrt{5x}}=\frac{5}{2\sqrt{5x}}$$

$$f(x)=x^3+x^2=\lim_{h\to 0}\frac{(x+h)^3+(x+h)^2-x^3-x^2}{h}=\lim_{h\to 0}3x^2+3xh+h+h^2+2x+h$$

$$=3x^2+2x$$

$$f(x)=\sqrt{x+1} \quad =\lim_{h\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\times\left(\frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}\right)$$

$$=\lim_{h\to 0}\frac{x+h+1-x-1}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}=\lim_{h\to 0}\frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$=\frac{1}{2\sqrt{x+1}}$$

$$f(x)=\frac{1}{x^2} \qquad \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x+h)^2}-\frac{1}{x^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{x^2-x^2-2xh-h^2}{h\left[x^2(x+h)^2\right]}$$

$$=\lim_{h\to 0}\frac{-2\cancel{x}-h}{x^{\cancel{x}}(x+h)^2}=\frac{-2}{x^3}$$

3. Determine the constant a so that the following function is continuous on the entire real line

$$f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$$

$a(2)^2 = 8$

$4a = 8$

$a = 2$

4. Find the average rate of change for each of the function over the given intervals:

(a) $y = x^2 + 1$ $[3,5]$

$\frac{f(5) - f(3)}{5-3}$

$\frac{26-10}{2} = 8$

(b) $f(x) = \frac{1}{x+1}$ $[0,3]$

$\frac{f(3) - f(0)}{3-0}$

$\frac{\frac{1}{4} - 1}{3} = -\frac{1}{4}$

5. Find the equations of the lines that are:
(i) tangent and (ii) normal to the curve

(a) $y = 5x^2$ at the point $(-1, 5)$

(i) $y' = 10x \rightarrow m = -10 \rightarrow y - 5 = -10(x+1) \rightarrow y = -10x - 5$

(ii) $y - 5 = \frac{1}{10}(x+1) \rightarrow y = \frac{1}{10}x + \frac{51}{10}$

(b) $y = x^2 - 4x$ at the point $(1, -3)$

(i) $y' = 2x - 4 \rightarrow y + 3 = -2(x-1) \rightarrow y = -2x - 1$

(ii) $y + 3 = \frac{1}{2}(x-1) \rightarrow y = \frac{1}{2}x - 7/2$

6. Using the symmetric difference quotient $\frac{f(a+h)-f(a-h)}{2h}$

Find : $f'(10)$ when $f(x)=x^3$ (Let $h=0.01$)

$$\frac{(a+h)^3-(a-h)^3}{2h} = \frac{(10+0.01)^3-(10-0.01)^3}{2(0.01)}$$

$$\approx \frac{1003-997}{0.02}$$

$$= 300$$

7. Find the first derivative of each of the following using the rules for differentiation.

a) $y = 5x^4 - 3x + \frac{4}{x} + 3 = 20x^3 - 3 + 4\ln x + 3x$

b) $y = \frac{3x}{2x+1} = \frac{3(2x+1)-3x(2)}{(2x+1)^2} = \frac{6x+3-6x}{(2x+1)^2} = \frac{3}{(2x+1)^2}$

c) $y = (2x^2+2)\sqrt{x} = 2x^{5/2} + 2x^{1/2} = 5x^{3/2} + x^{-1/2}$

d) $xy = \cos(x+y) =$

$$y + xy' = -\sin(x+y)(1+y')$$
$$= -\sin(x+y) - \sin(x+y)y'$$
$$xy' + \sin(x+y)y' = -\sin(x+y) - y$$
$$y' = \frac{-\sin(x+y)-y}{x+\sin(x+y)}$$

e) $y = x^3e^x + xe^{3x} = 3x^2e^x + x^3e^x + e^{3x} + 3xe^{3x}$

f) $y = 2^{5x} = (e^{\ln 2})^{5x} = 5(e^{\ln 2})^{5x} = \ln 5(2^{5x})$

g) $y = \log_3(\sqrt[3]{x}) = \frac{1}{3}\log_3 x = \frac{1}{3}\frac{\ln x}{\ln 3} \to y' = \frac{1}{3(\ln 3)x}$

$\log_3 x^{1/3}$

h) $y = \log_7(4 + x\ln 7)$ $y = \frac{\ln(4+x\ln 7)}{\ln 7} \to y' = \frac{1}{\ln 7} \cdot \frac{1}{4+x\ln 7} \cdot \ln 7 = \frac{1}{4+x\ln 7}$