Find dy/dx in problems 1-27.

1. $y = \dfrac{x}{\sqrt{x^2 - 4}}$

2. $x^2 + xy + y^2 - 5x = 2$

3. $x^2 y + yx^2 = 10$

4. $y = \cos(1 - 2x)$

5. $y = x^2\sqrt{x^2 - a^2}$

6. $y = \dfrac{x^2}{1 - x^2}$

7. $y = \sec^2(5x)$

8. $y = \dfrac{(2x^2 + 5x)^{\frac{3}{2}}}{3}$

9. $xy^2 + \sqrt{xy} = 2$

10. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

11. $(x + 2y)^2 + 2xy^2 = 6$

12. $y^2 = \dfrac{x}{x + 1}$

13. $x^3 - xy + y^3 = 1$

14. $y = \sqrt{\dfrac{1 + x}{1 - x}}$

15. $y = (x^3 + 1)^{\frac{1}{3}}$

16. $y = \cot 2x$

17. $y = \dfrac{\sin x}{\cos^2 x}$

18. $y = x^2 \cos 8x$

19. $y = \sin(\cos^2 x)$

20. $y = \sec^2 x$

21. $y = \sec x \sin x$

22. $y = \cos(\sin^2 x)$

23. $y = u^2 - 1, x = u^2 + 1$

24. $y = \sqrt{2t + t^2}, t = 2x + 3$

25. $t = \dfrac{x}{1 + x^2}, y = x^2 + t^2$

26. $x = t^2 - 1, y = 3t^4 - t^2$

27. $x = \cos 3t, y = \sin(t^2 + 1)$

28. Find the slope of $y = \dfrac{x}{x^2 + 1}$ at the origin. Write the equation of the tangent line at the origin.

29. Write the equation of the tangent at $(2,2)$ to the curve $x^2 - 2xy + y^2 + 2x + y - 6 = 0$.

30. What is the slope of the curve $y = 2x^2 - 6x + 3$ at the point on the curve where $x = 2$? Find the tangent to the curve at this point.

31. If $y = x\sqrt{2x - 3}$, find d^2y/dx^2.

32. Find the value of d^2y/dx^2 in the equation $y^3 + y = x$ at the point $(2,1)$.

33. Write an equation for the line through $(2,1)$ normal to the curve $x^2 = 4y$.

34. For what value of c is the curve $y = c/(x + 1)$ tangent to the line through the points $(0,3)$ and $(5,-2)$.

35. Show that the tangent to any point (a, a^3) on the curve $y = x^3$ meets the curve again at a point where the slope is 4 times the slope at (a, a^3).

36. Find the lines tangent and normal to the curve $(y - x)^2 = 2x + 4$ at the point $(6,2)$.

37. The circle $(x - h)^2 + (y - k)^2 = a^2$ is tangent to the curve $y = x^2 + 1$ at the point $(1,2)$.
a) Find the possible locations of the point (h,k).
b) If, in addition, the value of d^2y/dx^2 is the same on both curves at $(1,2)$, find h, k, and a.

Implicit differentiation Answer Key

(1) $y = \dfrac{x}{\sqrt{x^2-4}}$

$y = x(x^2-4)^{-1/2}$
$y' = u'v + v'u$
$y' = (x^2-4)^{-1/2} + -\frac{1}{2}(x^2-4)^{-3/2} \cdot 2x^2$
$y' = (x^2-4)^{-1/2} - x^2(x^2-4)^{-3/2}$
$y' = \dfrac{x^2-4-x^2}{\sqrt{(x^2-4)^3}}$

$y' = \dfrac{-4}{(x^2-4)^{3/2}}$

(2) $x^2 + xy + y^2 - 5x = 2$

$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} - 5 = 0$
$\dfrac{dy}{dx} = \dfrac{5-2x-y}{x+2y}$

(3) $x^2y + yx^2 = 10$

$2x^2y = 10$
$x^2y = 5$
$2xy + x^2\frac{dy}{dx} = 0$
$\dfrac{dy}{dx} = \dfrac{-2xy}{x^2}$

$\dfrac{dy}{dx} = \dfrac{-2y}{x}$

(4) $y = \cos(1-2x)$

$y' = -\sin(1-2x) \cdot -2$
$y' = 2\sin(1-2x)$

(5) $y = x^2\sqrt{x^2-a^2}$

$y' = u'v + v'u$
$y' = 2x\sqrt{x^2-a^2} + \frac{1}{2}(x^2-a^2)^{-1/2} \cdot 2x \cdot x^2$
$y' = (x^2-a^2)^{-1/2}(2x(x^2-a^2) + x^3)$

$y' = \dfrac{3x^3-2a^2x}{\sqrt{x^2-a^2}}$

(6) $y = \dfrac{x^2}{1-x^2}$

$y = x^2(1-x^2)^{-1}$
$y' = u'v + v'u$
$y' = 2x(1-x^2)^{-1} + -1(1-x^2)^{-2} \cdot -2x \cdot x^2$
$y' = 2x(1-x^2)^{-1} + 2x^3(1-x^2)^{-2}$
$y' = \dfrac{2x(1-x^2) + 2x^3}{(1-x^2)^2}$

$y' = \dfrac{2x}{(1-x^2)^2}$

(7) $y = \sec^2(5x)$
$y' = 2\sec(5x) \cdot \sec 5x \tan 5x \cdot 5$
$y' = 10\sec^2 5x \tan 5x$

(8) $y = \dfrac{(2x^2+5x)^{3/2}}{3}$

$y' = \frac{3}{2} \cdot \frac{1}{3}(2x^2+5x)^{1/2}(4x+5)$

$y' = \frac{1}{2}\sqrt{2x^2+5x}\,(4x+5)$

(9) $xy^2 + \sqrt{xy} = 2$

$y^2 + 2xy\frac{dy}{dx} + \frac{1}{2}(xy)^{-1/2}(y + x\frac{dy}{dx}) = 0$

$y^2 + 2xy\frac{dy}{dx} + \frac{1}{2}[x^{-1/2}y^{1/2} + \boxed{x^{1/2}y^{-1/2}}\frac{dy}{dx}] = 0$

$(2xy + \frac{1}{2}x^{1/2}y^{-1/2})\frac{dy}{dx} = -\frac{1}{2}x^{-1/2}y^{1/2} - y^2$

$\frac{dy}{dx} = \frac{-\frac{1}{2}x^{-1/2}y^{1/2} - y^2}{2xy + \frac{1}{2}x^{1/2}y^{-1/2}}$

(10) $x^{2/3} + y^{2/3} = a^{2/3}$

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x^{-1/3}}{y^{-1/3}} = \frac{-y^{1/3}}{x^{1/3}}$

(11) $(x+2y)^2 + 2xy^2 = 6$

$2(x+2y)(1+2\frac{dy}{dx}) + 2y^2 + 4xy\frac{dy}{dx} = 0$

$2x + 4x\frac{dy}{dx} + 4y + 8y\frac{dy}{dx} + 2y^2 + 4xy\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x-4y-2y^2}{4x+8y+4xy} = \frac{-x-2y-y^2}{2x+4y+2xy}$

(12) $y^2 = \frac{x}{x+1}$

$2y\frac{dy}{dx} = \frac{x+1-x}{(x+1)^2}$

$\frac{dy}{dx} = \frac{1}{2y(x+1)^2}$

(13) $x^3 - xy + y^3 = 1$

$3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}$

(14) $y = \sqrt{\frac{1+x}{1-x}}$

$y' = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2}\left(\frac{1-x+1+x}{(1-x)^2}\right)$

$y' = \frac{1}{(1-x)^{3/2}(1+x)^{1/2}}$

(15) $y = (x^3+1)^{1/3}$

$y' = \frac{1}{3}(x^3+1)^{-2/3}\cdot(3x^2)$

$y' = \frac{x^2}{(x^3+1)^{2/3}}$

(16) $y = \cot 2x$

$y' = -2\csc^2 2x$

(17) $y = \frac{\sin x}{\cos^2 x}$ $\frac{u}{v}$

$y' = \frac{\cos^3 x + 2\cos x \sin^2 x}{\cos^4 x}$

$y' = \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x}$

$y' = \frac{\cos^2 x + 2(1-\cos^2 x)}{\cos^3 x}$

$\to \frac{2-\cos^2 x}{\cos^3 x}$

$2\sec^3 x - \sec x$

(18) $y = x^2 \cos 8x$

$y' = u'v + v'u$

$y' = 2x\cos 8x - 8x^2 \sin 8x$

(19) $y = \sin(\cos^2 x)$

$y' = \cos(\cos^2 x)(-2\cos x \sin x)$

$y' = -\sin 2x \cos(\cos^2 x)$

(20) $y = \sec^2 x$

$y' = 2\sec^2 x \tan x$

(21) $y = \sec x \sin x$

$y' = u'v + v'u$

$y' = \sec x \tan x \sin x + \cos x \sec x$

$y' = \tan^2 x + 1$

(22) $y = \cos(\sin^2 x)$

$y' = -\sin(\sin^2 x)\cdot 2\sin x \cos x$

$y' = -\sin 2x \sin(\sin^2 x)$

(23) $y = u^2 - 1$, $x = u^2 + 1$

$y = x - 2$ $u^2 = x - 1$

$\frac{dy}{dx} = 1$

(24) $y=\sqrt{2t+t^2}$, $t=2x+3$

$y=\sqrt{4x+6+4x^2+12x+9}$

$y=\sqrt{4x^2+16x+15}$

$y'=\frac{1}{2}(4x^2+16x+15)^{-1/2}(8x+16)$

$y'=\dfrac{4x+8}{\sqrt{4x^2+16x+15}}$

(25) $t=\dfrac{x}{1+x^2}$, $y=x^2+t^2$

$y=x^2+\dfrac{x^2}{(1+x^2)^2}$

$y'=2x+\dfrac{2x(1+x^2)^2-2(1+x^2)\cdot 2x^3}{(1+x^2)^2}$

$y'=2x+\dfrac{2x+\cancel{4x^3}+2x^5-\cancel{4x^3}-4x^5}{(1+x^2)^2}$

$y'=2x+\dfrac{2x-2x^5}{(1+x^2)^4}$

(26) $x=t^2-1$, $y=3t^4-t^2$

$t^2=x+1 \qquad y=3(x+1)^2-x-1$

$y'=6(x+1)-1$

$y'=6x+5$

(27) $x=\cos 3t$, $y=\sin(t^2+1)$

$\dfrac{dx}{dt}=-3\sin 3t$

$\dfrac{dy}{dt}=2t\cos(t^2+1)$

$\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{2t\cos(t^2+1)}{-3\sin 3t}=\dfrac{dy}{dx}$

(28) $y=\dfrac{x}{x^2+1}$ @ $(0,0)$

$y'=\dfrac{x^2+1-2x^2}{(x^2+1)^2}$

$y'=\dfrac{1-x^2}{(x^2+1)^2}$ @ $(0,0)$

$y'=1 \Longrightarrow y=x$

(29) $x^2-2xy+y^2+2x+y-6=0$ @ $(2,2)$

$2x-2y-2x\frac{dy}{dx}+2y\frac{dy}{dx}+2+\frac{dy}{dx}=0$

$\dfrac{dy}{dx}=\dfrac{2y-2x-2}{-2x+2y+1}$ @ $(2,2)$

$\dfrac{dy}{dx}=\dfrac{-2}{1} \Longrightarrow y-2=-2(x-2)$

OR $y=-2x+6$

(30) $y=2x^2-6x+3$ @ $x=2$

$y'=4x-6$ @ $x=2$, $y'=2$

$y=2x+b$ @ $x=2$, $y=8-12+3=-1$

$-1=2(2)+b$

$y=2x-5$

(31) $y = x\sqrt{2x-3}$

$\frac{dy}{dx} = \sqrt{2x-3} + \frac{1}{2}(2x-3)^{-1/2}\cdot 2x$

$\frac{dy}{dx} = \sqrt{2x-3} + \frac{x}{\sqrt{2x-3}}$

$\frac{d^2y}{dx^2} = \frac{1}{2}(2x-3)^{-1/2}\cdot 2 + \frac{\sqrt{2x-3} - \frac{1}{2}(2x-3)^{-1/2}\cdot 2x}{2x-3}$

$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{2x-3}} + \frac{1}{\sqrt{2x-3}} - \frac{x}{(2x-3)^{3/2}}$

$\frac{d^2y}{dx^2} = \frac{2}{\sqrt{2x-3}} - \frac{x}{(2x-3)^{3/2}}$

$\frac{d^2y}{dx^2} = \frac{2(2x-3) - x}{(2x-3)^{3/2}}$

$\frac{d^2y}{dx^2} = \frac{3x-6}{(2x-3)^{3/2}}$

(32) $y^3 + y = x$ @ $(2,1)$

$3y^2\frac{dy}{dx} + \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{3y^2+1}$

$\frac{d^2y}{dx^2} = -1(3y^2+1)^{-2}\cdot 6y\,\frac{dy}{dx}$

$\frac{d^2y}{dx^2} = \frac{-6y}{(3y^2+1)^3}$ @ $(2,1)$

$\frac{d^2y}{dx^2} = \frac{-6}{4^3} = \frac{-6}{64} = \frac{-3}{32}$

(33) $x^2 = 4y$ @ $(2,1)$

$2x = 4\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{x}{2}$ @ $(2,1) \Rightarrow y' = 1$

Normal to curve: $y' = -1$

Eq: $y - 1 = -1(x-2)$

OR

$y = -x + 3$

(34) $y = \frac{c}{x+1}$

$\frac{dy}{dx} = \frac{-c}{(x+1)^2}$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-3}{5-0} = \frac{-5}{5} = -1$

$\frac{-c}{(x+1)^2} = -1 \Rightarrow \frac{c}{(x+1)^2} = 1$

$m = -1$ y-intercept $(0,3) \Rightarrow y = -x + 3$

$-x + 3 = \frac{c}{x+1} \Rightarrow c = (-x+3)(x+1)$

$\frac{c}{(x+1)^2} = 1 \Rightarrow c = (x+1)^2$

$(-x+3)(x+1) = (x+1)^2$

$-x + 3 = x + 1$

$2 = 2x$

$x = 1$

$\therefore\ c = (x+1)^2 = (1+1)^2 = 4$

* 37(b) $\frac{dy}{dx} = \frac{h-x}{y-k}$

$\frac{d^2y}{dx^2} = \frac{-(y-k)-y'(h-x)}{(y-k)^2}$

$\frac{d^2y}{dx^2} = \frac{-y-k-(h-x)^2/y-k}{(y-k)^2} = \frac{-2-k-\frac{(h-1)^2}{2-k}}{(2-k)^2} = 2$ plug in 'h' and 'k' solve for a then put back into h & k.

(35) $y = x^3$

$y' = 3x^2$ @ (a, a^3)

$y' = 3a^2$

$y' = 3x^2$

$4(3a^2) = 3x^2$

$12a^2 = 3x^2$

$4a^2 = x^2$

$2a = x$

$y = x^3 = (2a)^3 = 8a^3$

$(2a, 8a^3)$

$(2a, (2a)^3)$ if $b = 2a$ then

(b, b^3)

(36) $(y-x)^2 = 2x+4$ @ $(6,2)$

$2(y-x)(\frac{dy}{dx}-1) = 2$

$(y-x)(\frac{dx}{dx}-1) = 1$

$y\frac{dy}{dx} - y - x\frac{dy}{dx} + x = 1$

$\frac{dy}{dx} = \frac{1-x+y}{y-x} = \frac{1-6+2}{2-6} = \frac{-3}{-4}$

Tangent $m = 3/4$, normal $m_\perp = -4/3$

$y-2 = 3/4(x-6)$

OR

$y = 3/4x - \frac{18}{4} + 2$

$y = 3/4x - 5/2$

$y-2 = -4/3(x-6)$

OR

$y = -4/3x + 10$

(37) $(x-h)^2 + (y-k)^2 = a^2$

$y = x^2+1$, $\frac{dy}{dx} = 2x$

$(x-h)^2 + (x^2+1-k)^2 = a^2$

$2(x-h) + 2(y-k)\frac{dy}{dx} = 0$

$2(x-h) + 2(x^2+1-k)\cdot 2x = 0$

$2(x-h) + 4x(x^2+1-k) = 0$

@$(1,2) \Rightarrow 2(1-h) + 4(2-k) = 0$

$2(1-h) = -4(2-k)$

$1-h = -2(2-k)$

$h = -2(2-k)+1$

$h = 5-2k$

$(x-h)^2 + (y-k)^2 = a^2$

$(1-5+2k)^2 + (2-k)^2 = a^2$

$(2k-4)^2 + (2-k)^2 = a^2$

$4k^2-16k+16+4-4k+k^2 = a^2$

$5k^2 - 20k + 20 = a^2$

$5k^2 - 20k + (20-a^2) = 0$

$k = \frac{20 \pm \sqrt{(-20)^2 - 4(5)(20-a^2)}}{10}$

$k = \frac{20 \pm \sqrt{400-400+20a^2}}{10}$

$k = \frac{20 \pm 2\sqrt{5}a}{10} \Rightarrow k = \frac{10 \pm \sqrt{5}a}{5}$

$h = 5 - 2\left(\frac{10 \pm \sqrt{5}a}{5}\right)$

$h = 5 - \frac{20 \pm 2\sqrt{5}a}{5}$

$h = \frac{5 \pm 2\sqrt{5}a}{5}$